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The fundamental matrix in three-dimensional dissipative gasdynamics

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Three-dimensional steady flow past a body placed in a uniform stream of viscous, thermally conducting fluid is considered within the framework of the Oseen approximation. Asymptotic forms for the fundamental matrix are obtained for both supersonic and subsonic flow. It is shown how the solution to the flow past a body may be obtained from the fundamental matrix, and that the fundamental matrix itself provides the far field flow.

1. Introduction

The fundamental matrix for steady three-dimensional flow of a viscous, heat-conducting, compressible fluid is studied within the framework of the Oseen linearization. An asymptotic form of the solution is obtained, which is applicable for large distances from the origin compared with the characteristic length of the fluid (which is of the order of the mean free path).

It is shown how the problem of the flow past bodies can be formulated in terms of the fundamental matrix. These problems lead to integral equations, the solutions of which determine the manner in which the fundamental matrix is distributed to yield the desired flow field. Because of the complexity of the resulting integral equations, this has not been carried out in any specific case. However, it is shown that the fundamental matrix itself furnishes the flow directly for large distance from the body compared with body size. The solution is given in terms of such quantities as the total force on the body and the total heat added to the fluid by the body.

The effect of viscosity on compressible fluid flow was investigated by Lagerstrom, Cole & Trilling (1949), in the study of problems of wave propagation and of two-dimensional steady flows. Fundamental solutions for these problems were considered, and asymptotic forms were derived for the various cases. These solutions exhibit the viscous 'quasi-wave' noted by earlier workers (see the paper cited above for a bibliography), which becomes the hyperbolic wave in the limit of vanishing viscosity. Thus the inviscid characteristics play a central role: while they no longer represent discontinuities, for large values of the local Reynolds number they locate regions of rapid but continuous change. Therefore viscosity acts to smooth out the discontinuities of the inviscid theory. Accordingly, in obtaining the fundamental matrix the point of view taken in the present paper is that the inviscid theory provides an adequate description of the flow field except in the neighbourhood of its discontinuities, the characteristics. These are the wake and, in the supersonic case, the Mach cone. These singularities indicate regions where the dissipation becomes important, and a boundary-layer-type analysis is carried out in their neighbourhood. The corresponding two-dimensional analysis was carried out by Sirovich (1968); the present paper represents an extension of this work to three dimensions.

For any problem involving fundamental solutions, a linear set of equations is required. The linearization used here, as in the two-dimensional steady flow problems mentioned above, corresponds to the Oseen linearization (i.e. the flow variables are considered to be small perturbations of a uniform free-stream flow). This approximation is, of course, not valid near the body, where the velocity vanishes. In fact, the Oseen equations give a qualitatively incorrect description of the boundary layer. However, the linearization should be applicable far from the body, since all disturbances caused by the body eventually die out. Thus the Oseen equations can be used to describe the structure of the viscous wake far from the body.

However, difficulties arise in using the linearized equations to describe supersonic flow. Consider inviscid flow past a slender body. According to the linear theory, Mach lines (or Mach cones in three dimensions) emerge from the body, and are inclined at an angle determined by the free-stream Mach number. In the non-linear theory, however, these Mach lines are not straight and parallel, but are curved, and shocks emerge from the leading and trailing edge. Also, the flow behind the rear shock is not uniform. Even for very slender bodies this pattern emerges as one goes far enough from the body, since the effects of the non-linear terms, while small, are cumulative. Thus the linear theory is a non-uniform approximation, which cannot describe the neighbourhood of infinity.

For two-dimensional flow, the effects of the non-linearity can be accounted for to a first approximation by the simple wave theory (Friedrichs 1948). Whitham (1952) obtains a uniformly valid first approximation for bodies of revolution, using as a basis essentially the method of strained co-ordinates (see e.g. Van Dyke 1964). In these flows, the far field exhibits the 'N wave', in which the pressure jumps discontinuously across the front shock, then decreases linearly, until it jumps again across the tail shock. In the linear theory, the shock waves do not appear, the spreading of the characteristics does not occur, and the theory cannot describe the approach to infinity. However, for sufficiently slender bodies, there is a large region of the flow field which can be adequately described by the linear theory. In addition, the linear theory provides a good model in which the effects of viscosity and thermal conductivity on compressible fluid flow can be studied in the general three-dimensional case.

2. The role of the fundamental matrix

Asymptotic fundamental solutions for two-dimensional steady gasdynamics have been obtained by Sirovich (1961, p. 283) for both supersonic and subsonic flow. A technique which shows how boundary-value problems can be formulated in terms of the fundamental solutions (or, equivalently, the fundamental matrix) has been developed by Sirovich and Salathe, and applied to a number of problems in gasdynamics and magnetohydrodynamics (Salathe & Sirovich 1967, 1968; Sirovich 1967, 1968).

Sirovich (1968) discusses the steady dissipative gasdynamic flow past two-dimensional bodies. The formulation of the problem in terms of the fundamental solution, and its role in the description of the far field, was carried out for an arbitrary number of space dimensions. A brief review of that discussion will be given, and the remainder of the paper will be devoted to obtaining the fundamental matrix.

If the domain of definition of the dependent variables is extended over all space by defining a source-free flow in the interior of the body, then the body surface, denoted by $\hat{S}(\hat{\mathbf{x}}) = 0$, becomes a surface of discontinuity. It can be shown (Sirovich 1967, appendix 1) that, assuming no flow across the body surface, the governing equations are

$$\left.\begin{aligned} &\hat{\nabla}.\hat{\rho}\hat{\mathbf{u}} = 0,\\ &\boldsymbol{\nabla}.(\hat{\rho}\hat{\mathbf{u}}\hat{\mathbf{u}}+\hat{p}\mathbf{1}-\hat{\boldsymbol{\Pi}}) = [\hat{p}\mathbf{1}-\hat{\boldsymbol{\Pi}}].\mathbf{n}\delta(\hat{S})\\ &\hat{\boldsymbol{\nabla}}.[\hat{\rho}\hat{\mathbf{u}}\,(e+\tfrac{1}{2}\hat{u}^2)+\hat{p}\hat{\mathbf{u}}-\hat{\boldsymbol{\Pi}}.\hat{\mathbf{u}}+\hat{\mathbf{Q}}] = [\hat{\mathbf{Q}}].\mathbf{n}\delta(\hat{S})\\ &\hat{\Pi}_{ij} = \mu(\hat{u}_{i,j}+\hat{u}_{j,i})+(\beta-\tfrac{2}{3}\mu)\,\hat{\boldsymbol{\nabla}}.\hat{\mathbf{u}}\delta_{ij}\\ &\hat{\mathbf{Q}}+\kappa\hat{\boldsymbol{\nabla}}\hat{T} = \kappa[\hat{T}]\mathbf{n}\delta(\hat{S}), \end{aligned}\right\} \tag{2.1}$$

where $\hat{\rho}$, $\hat{\mathbf{u}}$, e, $\hat{p}$, $\hat{T}$ are the fluid density, velocity, internal energy, pressure and temperature; $\hat{\boldsymbol{\Pi}}$, $\hat{\mathbf{Q}}$, μ, β, κ the stress tensor, heat flow vector, and coefficients of viscosity, bulk viscosity and heat conductivity; and $\mathbf{1}$ denotes the unit matrix. $\delta(\hat{S})$ is the delta function defined on $\hat{S}(\hat{\mathbf{x}}) = 0$, and can be written

$$\delta(\hat{S}) = \int_{\hat{S}} \prod_{i=1}^{3} \delta(\hat{x}_i - \hat{x}'_i)\, d\hat{S}(\hat{\mathbf{x}}'). \tag{2.2}$$

It has the property
$$\int_V f(\hat{\mathbf{x}})\,\delta(\hat{S})\,dV = \int_{\hat{S}} f\,d\hat{S},$$

where V is any volume containing $\hat{S}$, and $f(\hat{\mathbf{x}})$ is any function defined over V. The square brackets [ ] denote the jump across the surface $\hat{S} = 0$, and $\mathbf{n}$ is the outward drawn normal to this surface.

Oseen equations

We consider steady flows linearized about the uniform upstream conditions, and define the following normalized perturbation quantities:

$$\left.\begin{aligned} &\rho = \frac{\hat{\rho}-\rho_0}{\rho_0}, \quad \mathbf{u} = \frac{\hat{\mathbf{u}}-\mathbf{U}_0}{a_0}, \quad T = \left(\frac{c_v}{T_0 a_0^2}\right)^{\frac{1}{2}}(\hat{T}-T_0),\\ &p = \frac{\hat{p}-p_0}{\rho_0 a_0^2}, \quad P = \frac{\hat{p}}{\rho_0 a_0^2} = P_0 + p, \quad \mathbf{U} = \frac{\mathbf{U}_0}{a_0},\\ &\boldsymbol{\Pi} = \frac{\hat{\boldsymbol{\Pi}}}{\rho_0 a_0^2}, \quad \mathbf{Q} = \frac{\hat{\mathbf{Q}}}{\rho_0 a_0^2 (c_v T_0)^{\frac{1}{2}}}, \quad \mathbf{x} = \frac{\hat{\mathbf{x}}}{L}, \end{aligned}\right\} \tag{2.3}$$

where zero subscript denotes the free-stream value and

$$a_0 = ((\partial p_0/\partial \rho_0)_{T_0})^{\frac{1}{2}}, \quad c_v = (\partial e_0/\partial T_0)_{\rho_0}.$$

L is an unspecified length scale, which we shall later make definite. Taking the complimentary flow to be identically zero, the linearized equations are:

$$\left.\begin{aligned} &\mathbf{U}.\nabla\rho+\nabla.\mathbf{u} = 0,\\ &\mathbf{U}.\nabla\mathbf{u}+\nabla\rho+\chi\nabla T-\zeta\nabla^2\mathbf{u}-\eta\nabla\nabla.\mathbf{u} = \mathbf{n}.(P\mathbf{1}-\mathbf{\Pi})\,\delta(S),\\ &\mathbf{U}.\nabla T+\chi\nabla.\mathbf{u}+\nabla.\mathbf{Q} = \mathbf{n}.\mathbf{Q}\delta(S)-\overline{\chi}\mathbf{U}.(P\mathbf{n}-\mathbf{\Pi}.\mathbf{n})\,\delta(S),\\ &\mathbf{Q} = -\xi\nabla T+\xi\mathbf{n}T\delta(S),\end{aligned}\right\} \tag{2.4}$$

where S denotes the body surface in terms of the normalized co-ordinates, and

$$\left.\begin{aligned} &\eta = (\beta+\tfrac{1}{3}\mu)/(\rho_0 a_0 L), \quad \zeta = \mu/(a_0\rho_0 L),\\ &\xi = \kappa/(\rho_0 a_0 c_v L), \quad \chi = (\gamma-1)^{\frac{1}{2}}, \quad \overline{\chi} = \frac{a_0}{(c_v T_0)^{\frac{1}{2}}}.\end{aligned}\right\} \tag{2.5}$$

We have used the relationship,

$$\gamma = \frac{c_p}{c_v} = \frac{c_0^2}{a_0^2},$$

where

$$c_0^2 = (\partial p_0/\partial \rho_0)_{S_0},$$

and c_p is the specific heat at constant pressure. The equations of state of the gas, $e = e(\hat{\rho},\hat{T})$, $\hat{p} = \hat{p}(\hat{\rho},\hat{T})$, have been left arbitrary subject only to the compatibility relation,

$$\hat{p}-\hat{\rho}^2(\partial e/\partial\hat{\rho})_{\hat{T}} = \hat{T}(\partial\hat{p}/\partial\hat{T})_{\hat{\rho}}.$$

In obtaining (2.4), the momentum equation was used to subtract the mechanical work from the energy equation.

We note that the sources in the above equations are given in terms of the stress, heat conduction and temperature at the body surface. The equations can be written in the condensed form

$$\mathbf{L}\mathbf{w} = \mathbf{F}\delta(S). \tag{2.6}$$

Thin bodies

Suppose the body is thin and can be written in the form

$$z = \epsilon f^{\pm}(x,y),$$

where ϵ is a small number. Then the source terms in (2.6) can be simplified by expanding in powers of ϵ (Sirovich 1968). Let $\theta(x,y)$ be the characteristic function of the projection S^* of S onto $z = 0$; i.e. $\theta = 1$ if $(x,y)\in S^*$, $\theta = 0$ otherwise. Then

$$\mathbf{F}\delta(S) = \theta(x,y)\{(\mathbf{F}^+ +\mathbf{F}^-)\,\delta(z)-\epsilon\delta'(z)\,(\mathbf{F}^+f^+ +\mathbf{F}^-f^-)\}+O(\epsilon^2),$$

where $\mathbf{F}^{\pm}$ is the value of $\mathbf{F}$ at the upper or lower surface. Using

$$\mathbf{n}^{\pm} = \pm\mathbf{e}_z \mp \epsilon\nabla f^{\pm}+O(\epsilon^2),$$

where $\mathbf{e}_z$ is the unit vector in the z direction, we find:

$$\begin{aligned}\mathbf{F}^{\pm} &= [0,\ \pm\mathbf{e}_z.(P^{\pm}\mathbf{1}-\mathbf{\Pi}^{\pm}),\ \pm\mathbf{e}_z.(\mathbf{Q}^{\pm}-\bar{\chi}\mathbf{U}.\{P^{\pm}\mathbf{1}-\mathbf{\Pi}^{\pm}\},\ \pm\xi\mathbf{e}_z T^{\pm}]\\ &\quad+\epsilon[0,\ \mp(P^{\pm}\mathbf{1}-\mathbf{\Pi}^{\pm}).\nabla f^{\pm},\ \mp(\mathbf{Q}^{\pm}-\bar{\chi}\mathbf{U}.\{P^{\pm}\mathbf{1}-\mathbf{\Pi}^{\pm}\}).\nabla f^{\pm},\ \mp\xi T^{\pm}\nabla f^{\pm}]\\ &\quad+O(\epsilon^2)\\ &= \mathbf{F}_0^{\pm}+\epsilon\mathbf{F}_1^{\pm}+O(\epsilon^2). \end{aligned}\tag{2.7}$$

Therefore we have to first order in ϵ:

$$\begin{aligned}\mathbf{F}\delta(S) &= \theta(x,y)\{(\mathbf{F}_0^{+}+\mathbf{F}_0^{-})\,\delta(z)+\epsilon(\mathbf{F}_1^{+}+\mathbf{F}_1^{-})\,\delta(z)\\ &\quad-\epsilon\delta'(z)\,(\mathbf{F}_0^{+}f^{+}+\mathbf{F}_0^{-}f^{-})\}+O(\epsilon^2)\\ &= \theta(x,y)\{\mathbf{N}(x,y)\,\delta(z)+\mathbf{N}^*(x,y)\,\delta'(z)\}+O(\epsilon^2).\end{aligned}\tag{2.8}$$

Consequently, for thin bodies, (2.6) can be written in the form,

$$\begin{aligned}\mathbf{Lw} &= \delta(z)\iint\mathbf{N}(x',y')\,\delta(x-x')\,\delta(y-y')\,\theta(x',y')\,dx'\,dy'\\ &\quad+\delta'(z)\iint\mathbf{N}^*(x',y')\,\delta(x-x')\,\delta(y-y')\,\theta(x',y')\,dx'\,dy'.\end{aligned}\tag{2.9}$$

The fundamental matrix† $\mathbf{W}$ is defined as the solution to

$$\mathbf{LW} = \mathbf{1}\delta(\mathbf{x}),$$

where $\mathbf{1}$ denotes the unit matrix.

From (2.9) it is evident that the solution $\mathbf{w}$ is given in terms of the fundamental matrix by

$$\begin{aligned}\mathbf{w} &= \iint\mathbf{W}(x-x',y-y',z).\mathbf{N}(x',y')\,\theta(x',y')\,dx'\,dy'\\ &\quad+\frac{\partial}{\partial z}\iint\mathbf{W}(x-x',y-y',z).\mathbf{N}^*(x',y')\,\theta(x',y')\,dx'\,dy'.\end{aligned}\tag{2.10}$$

We note that the zeroth-order solution in ϵ corresponds to the flow past a flat plate with characteristic function $\theta(x,y)$.

Far field

We have seen that for thin bodies the solution is given as an integration of the fundamental matrix over a portion of the (x,y) plane. It will now be shown that, for distances from the body which are large compared with body size, the fundamental matrix itself (or, more precisely, the fundamental solution) furnishes the flow field directly. To see this, we normalize length with respect to distance R from the body. Then, for R large compared with a characteristic body dimension l, we have

$$\begin{aligned}\mathbf{F}\delta(S) &= \int_S\mathbf{F}(\mathbf{x}_0)\,\delta(\mathbf{x}-\mathbf{x}_0)\,dS_0\\ &= \delta(\mathbf{x})\int_S\mathbf{F}\,dS_0-\frac{\partial}{\partial x_k}\delta(\mathbf{x})\int_S x_{0k}\mathbf{F}\,dS_0+\dots\\ &= \mathbf{G}\delta(\mathbf{x})+O\left(\frac{l}{R}\right),\end{aligned}\tag{2.11}$$

which represents a multipole expansion, obtained by expanding $\delta(\mathbf{x}-\mathbf{x}_0)$.

† This is often referred to as the Green's Tensor.

Consequently, (2.6) reduces to the equations for the fundamental solution,

$$\mathbf{L}\mathbf{W}^f = \mathbf{G}\delta(\mathbf{x}),$$

where $\mathbf{G}$, the source strength, is a constant. The fundamental solution is related to the fundamental matrix by $\mathbf{W}^f = \mathbf{W}.\mathbf{G}$. Equation (2.11) shows that the source strengths $\mathbf{G}$ are the quantities $\mathbf{F}$ integrated over the body surface. In the momentum equation, this is the total pressure and viscous force exerted on the fluid by the body; in the energy equation, it is the total heat added by the body. The source in the heat conduction equation will be referred to as the virtual heat conduction; it is significant when, for example, a temperature gradient is maintained across the body.

It should be emphasized that this result is not based on the assumption of a thin body. However, the linearization is essential in all the above, since the mechanical work was subtracted from the energy equation by multiplying the momentum equation by $\mathbf{U}$. Without the linearization, the momentum equation must be multiplied by $\mathbf{u}$, which involves a product of distributions, a meaningless concept.

Reduction to five equations

The above set of equations can be reduced to five equations by substituting $\mathbf{Q}$ from the heat conduction equation into the energy equation. We shall consider the following system for the fundamental matrix:

$$\mathbf{A}\mathbf{V} = \mathbf{1}\delta(\mathbf{x}), \tag{2.12}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{U}.\nabla & \nabla. & 0 \\ \nabla & \mathbf{U}.\nabla - \zeta\nabla^2 - \eta\nabla\nabla. & \chi\nabla \\ 0 & \chi\nabla. & \mathbf{U}.\nabla - \xi\nabla^2 \end{bmatrix}. \tag{2.13}$$

The zeroth-order solution for flow past a thin body (corresponding to flow past a disk) is given in terms of $\mathbf{V}$ by†

$$\mathbf{v} = \iint \theta(x', y')\,\mathbf{V}(x-x', y-y', z).\begin{bmatrix} 0 \\ \mathbf{e}_z.[P\mathbf{1} - \mathbf{\Pi}] \\ \mathbf{e}_z.[\mathbf{Q} - \overline{\chi}(P\mathbf{1} - \mathbf{\Pi}).\mathbf{U}] \end{bmatrix} dx'\,dy', \tag{2.14}$$

(here [ ] denotes the jump across the body) and the far field flow by

$$\mathbf{v} \sim \mathbf{V}.\int_S [0, \mathbf{n}.(P\mathbf{1} - \mathbf{\Pi}), \mathbf{n}.\mathbf{Q} - \overline{\chi}\mathbf{n}.(P\mathbf{1} - \mathbf{\Pi}).\mathbf{U}]\,dS$$
$$+ \left\{\int_S \mathbf{n}\xi T\,ds\right\}.\nabla\mathbf{V}.[0,0,0,0,1]\,dS, \tag{2.15}$$

where $\mathbf{v} = \{\rho, \mathbf{u}, T\}$.

3. Non-dissipative theory

In §4, we shall obtain the fundamental solution corresponding to the Oseen operator (2.12). It has already been shown that such a solution furnishes the

† To this order the source term $\xi\mathbf{e}_z[T]$ in the heat conduction equation can be ignored, since $[T] = O(\epsilon)$ under usual circumstances.

flow far from a body. The method of solution is essentially a boundary-layer analysis carried out in the regions of singularities of the non-dissipative solution. Consequently, the non-dissipative fundamental matrix must first be determined.

The non-dissipative fundamental matrix is the solution to

$$\mathbf{A}_{ND}\mathbf{V} = \mathbf{1}\delta(\mathbf{x}), \tag{3.1}$$

where

$$\mathbf{A}_{ND} = \begin{bmatrix} \mathbf{U}.\nabla & \nabla. & 0 \\ \nabla & \mathbf{U}.\nabla & \chi\nabla \\ 0 & \chi\nabla. & \mathbf{U}.\nabla \end{bmatrix}. \tag{3.2}$$

To solve (3.1), we introduce Fourier transforms. Then

$$\mathbf{A}_{ND}(i\mathbf{k}) = \begin{bmatrix} ik_1U & ik_1 & ik_2 & ik_3 & 0 \\ ik_1 & ik_1U & 0 & 0 & ik_1\chi \\ ik_2 & 0 & ik_1U & 0 & ik_2\chi \\ ik_3 & 0 & 0 & ik_1U & ik_3\chi \\ 0 & ik_1\chi & ik_2\chi & ik_2\chi & ik_1U \end{bmatrix}, \tag{3.3}$$

so that $\mathbf{V}(i\mathbf{k}) = \mathbf{A}_{ND}^{-1}$, and

$$\begin{aligned} \mathbf{V}(\mathbf{x}) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} e^{i\mathbf{k}.\mathbf{x}}\mathbf{A}_{ND}^{-1}(i\mathbf{k})\,d\mathbf{k}, \\ &= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} \frac{e^{i\mathbf{k}.\mathbf{x}}\hat{\mathbf{C}}_{ND}\,d\mathbf{k}}{D_{ND}}, \end{aligned} \tag{3.4}$$

where D_{ND} is the determinant of $\mathbf{A}_{ND}(i\mathbf{k})$ and $\hat{\mathbf{C}}_{ND}$ the transposed cofactor matrix. The determinant is given by $D_{ND} = ik_1^3U^3\{k_1^2(U^2-\gamma)-\gamma(k_2^2+k_3^2)\}$, and $\hat{\mathbf{C}}_{ND}$ can be written $\hat{\mathbf{C}}_{ND} = k_1^2U^2\mathbf{C}_{ND}$, where the elements of $\mathbf{C}_{ND}$ are polynomials in ik_1, ik_2, ik_3. They can be extracted from the integration as derivatives to give:†

$$\mathbf{V}(\mathbf{x}) = \mathbf{C}_{ND}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Phi(x,y,z), \tag{3.5}$$

where

$$\left.\begin{aligned} \Phi &= \frac{1}{(2\pi)^3\Gamma}\int_{-\infty}^{+\infty}\frac{e^{i\mathbf{k}.\mathbf{x}}d\mathbf{k}}{k_1\left[k_1^2-\dfrac{\gamma}{U^2-\gamma}(k_2^2+k_3^2)\right]} \\ \text{and}\qquad \Gamma &= iU(U^2-\gamma). \end{aligned}\right\} \tag{3.6}$$

We carry out the integration of (3.6) first with respect to k_1. If $(U^2-\gamma) > 0$, corresponding to supersonic flow, $D = 0$ has three real roots. If $U^2-\gamma < 0$, corresponding to subsonic flow, then $D = 0$ has one real root, $k_1 = 0$, and two complex conjugate roots. The path of integration is deformed under the real roots because, as shown in §4, the effect of dissipation is to move them into the upper half-plane.

† By $\mathbf{C}_{ND}(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ we mean a matrix operator obtained from $\mathbf{C}_{ND}(ik_1, ik_2, ik_3)$ by replacing ik_1, ik_2, ik_3 by $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ respectively.

Denoting by Φ_0 the contribution to Φ from the $k_1 = 0$ root, and by Φ_h or Φ_e the remaining contribution in the supersonic (hyperbolic) or subsonic (elliptic) case, we obtain

$$\Phi_0 = \frac{1}{2\pi} H(x) \frac{1}{\gamma U} \ln (y^2+z^2)^{\frac{1}{2}}, \tag{3.7}$$

$$\Phi_h = -\frac{1}{2\pi} H(x) \frac{1}{\gamma U} \cosh^{-1} \frac{\alpha_h x}{(y^2+z^2)^{\frac{1}{2}}} - \frac{1}{2\pi} H(x) \frac{1}{\gamma U} \ln (y^2+z^2)^{\frac{1}{2}}, \tag{3.8}$$

$$\Phi_e = -\frac{1}{4\pi\gamma U} \sinh^{-1} \frac{\alpha_e x}{(y^2+z^2)^{\frac{1}{2}}} - \frac{1}{4\pi\gamma U} \ln (y^2+z^2)^{\frac{1}{2}} \operatorname{sgn}(x), \tag{3.9}$$

where

$$\alpha_h^2 = \frac{\gamma}{U^2-\gamma} = -\alpha_e^2.$$

Φ_0 can be added to Φ_h or Φ_e producing a cancellation (or partial cancellation in the subsonic case) of the logarithmic terms. However, as is well known, the three-dimensional solution contains a wake which, e.g. exhibits across it a jump in u_y. In the present form, these effects are contained in Φ_0; Φ_h and Φ_e do not contribute to the wake. Consequently, when the dissipative solution is discussed, it will be seen that it is necessary to refine Φ_0 in the neighbourhood of $y^2+z^2 = 0$, while Φ_h must be altered only along the Mach cone $\alpha_h^2 x^2 - (y^2+z^2) = 0$, and Φ_e remains unchanged.

In addition, the elements of $\mathbf{C}_{ND}$ take particularly simple forms when operating on either Φ_0 or Φ_e and Φ_h. In fact, we can write (3.5) in the form

$$V_{ij}(\mathbf{x}) = -X_i\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) X_j\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \Phi_{h,e}(x,y,z) + C_{ij}^0\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \Phi_0, \tag{3.10}$$

where

$$X = \left(\frac{U}{\gamma^{\frac{1}{2}}} \frac{\partial}{\partial x}, -\gamma^{\frac{1}{2}} \frac{\partial}{\partial x}, -\gamma^{\frac{1}{2}} \frac{\partial}{\partial y}, -\gamma^{\frac{1}{2}} \frac{\partial}{\partial z}, \left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{2}} U \frac{\partial}{\partial x}\right), \tag{3.11}$$

and

$$\left.\begin{aligned} C_{11}^0 &= (\gamma-1)\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = -(\gamma-1)^{\frac{1}{2}} C_{15}^0 = \frac{\gamma-1}{\gamma} C_{22}^0 = (\gamma-1) C_{55}^0, \\ C_{33}^0 &= \gamma \frac{\partial^2}{\partial z^2}, \quad C_{34}^0 = -\gamma \frac{\partial^2}{\partial y \, \partial z}, \quad C_{44}^0 = \gamma \frac{\partial^2}{\partial y^2}. \end{aligned}\right\} \tag{3.12}$$

The reason for the existence of this representation is the following: the contributions to the integral (3.6) occur only at the zeros of D. $\mathbf{C}_{ND}$ is therefore the classical adjoint of a degenerate matrix, and for the Φ_h and Φ_e contribution its rank is $n-1$. From this (3.11) follows (see e.g. Nomizu 1966). For the Φ_0 contribution, the rank is $n-3$, and such a representation does not exist. However, (3.12) follows, since $\partial\Phi_0/\partial x = 0$.

Flow past bodies

Equation (2.10) can be used to obtain the standard solutions of three-dimensional wing theory. For non-dissipative flows the heat conduction equation does not enter, and **w**, **W** in (2.10) can be formally replaced by **v** and **V**, where **V** is given by (3.7)–(3.12). The vectors **N**, **N*** can be replaced by the 5-vectors,

$$\mathbf{N} = \{0, -P_0(f_x^+ - f_x^-), -P_0(f_y^+ - f_y^-), (p_u - p_l), P_0 U\bar{\chi}(f_x^+ - f_x^-)\}$$

$$\mathbf{N}^* = \{0, 0, 0, -P_0(f^+ - f^-), 0\}.$$

These are obtained from (2.7) and (2.8), when $\xi = \eta = \zeta = 0$.

4. Dissipative theory

In this section, the fundamental solution corresponding to the dissipative equations will be obtained. As pointed out in §2, the fundamental solution provides the far field solution. It also furnishes the full flow field if the superposition described in §2 is carried out. This, however, will not be done in the present paper.

The equations for the fundamental solution are given above ((2.12), (2.13)), and, as in the non-dissipative case, the solution can be written in the form,

$$\mathbf{V}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{A}^{-1}(i\mathbf{k})\,d\mathbf{k}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\mathbf{C}}\,d\mathbf{k}}{D}, \tag{4.1}$$

where $\mathbf{A}^{-1}$ denotes the inverse of **A**, and $\hat{\mathbf{C}}$ and **D** are the transposed cofactor matrix and determinant of **A**, respectively.

The integration indicated in (4.1) cannot be carried out, and so an asymptotic solution will be obtained. The point of view taken is that the non-dissipative solution is satisfactory in the regions bounded away from its singularities, which occur along the characteristics. In the present case, these are the wake and the Mach cone. They indicate regions where the higher-order derivatives, or the dissipative terms, become important. We shall examine the neighbourhood of the characteristics using a boundary-layer-type analysis.

The wake†

The wake is characterized by the fact that the derivatives across it are large compared with the derivatives along it. If the previously unspecified length scale L is fixed by

$$\max\{\zeta, \eta, \xi\} = 1, \tag{4.2}$$

then the condition on the derivatives becomes, in terms of the wave-numbers,

$$k_1 \ll (k_2, k_3) \ll 1. \tag{4.3}$$

It is then found that the leading terms in D are:

$$D \sim [\gamma i k_1 U + \xi(k_2^2 + k_3^2)][ik_1 U + \zeta(k_2^2 + k_3^2)]^2 (k_2^2 + k_3^2). \tag{4.4}$$

† The asymptotic analysis is carried out in wave-number space, following and generalizing to three dimensions a method given by Sirovich (1961).

This result was obtained by requiring the leading dissipative and non-dissipative terms to be of the same order. A consequence of this is that

$$k_1 \sim k_2^2, k_3^2, \tag{4.5}$$

which furnishes the relative order of k_1, k_2, k_3.

With this ordering, it can readily be found that the non-zero $\hat{C}_{ij}$ are to lowest order:

$$\hat{C}_{11} = -\sqrt{(\gamma-1)}\,\hat{C}_{15} = (\gamma-1)\,\hat{C}_{55} = [ik_1U + \zeta(k_2^2+k_3^2)]^2(k_2^2+k_3^2)(\gamma-1),$$

$$\hat{C}_{22} = \frac{k_2^2+k_3^2}{k_3^2}\hat{C}_{33} = \frac{k_2^2+k_3^2}{k_2^2}\hat{C}_{44} = -\frac{(k_2^2+k_3^2)}{k_2k_3}\hat{C}_{34}$$

$$= [ik_1U + \zeta(k_2^2+k_3^2)]\,[\gamma ik_1U + \xi(k_2^2+k_3^2)]\,(k_2^2+k_3^2)$$

(by symmetry, $\hat{C}_{ij} = \hat{C}_{ji}$). From this we can deduce that

$$\left.\begin{aligned}
V_{11} &= \frac{\gamma-1}{4\pi x\xi}\exp\left(-\frac{\gamma U(y^2+z^2)}{4x\xi}\right) = -(\gamma-1)^{\frac{1}{2}}V_{15} = (\gamma-1)\,V_{55},\\
V_{22} &= \frac{1}{4\pi x\zeta}\exp\left(-\frac{U(y^2+z^2)}{4x\zeta}\right),\\
(V_{33}, V_{44}) &= \frac{(z^2,y^2)}{4\pi(y^2+z^2)\,x\zeta}\exp\left(-\frac{U(y^2+z^2)}{4x\zeta}\right)\\
&\quad + \left(\frac{1}{2\pi U(y^2+z^2)} - \frac{(z^2,y^2)}{\pi U(y^2+z^2)^2}\right)\left\{1-\exp\left(-\frac{U(y^2+z^2)}{4x\zeta}\right)\right\},\\
V_{34} &= -\frac{yz}{4\pi(y^2+z^2)\,x\zeta}\exp\left(-\frac{U(y^2+z^2)}{4x\zeta}\right)\\
&\quad + \frac{yz}{\pi U(y^2+z^2)}\left\{1-\exp\left(-\frac{U(y^2+z^2)}{4x\zeta}\right)\right\}.
\end{aligned}\right\} \tag{4.6}$$

These solutions are the extension to the dissipative case of that portion of the non-dissipative solution given by Φ_0.

The entire solution is given by adding the contribution from Φ_E or Φ_h, although, as remarked in §3 and shown below, Φ_h is also altered by the presence of dissipation.

In the limit ζ, $\xi \to 0$ the above solutions reduce to

$$\left.\begin{aligned}
V_{11} &= \frac{\gamma-1}{\gamma U}\delta(y)\,\delta(z) = -(\gamma-1)^{\frac{1}{2}}V_{15} = (\gamma-1)\,V_{55},\\
V_{22} &= \delta(y)\,\delta(z),\\
V_{ij} &= \delta(y)\,\delta(z)\,[\delta_{i3}\,\delta_{j3}\,z^2 + \delta_{i4}\,\delta_{j4}\,y^2 - \delta_{i3}\,\delta_{j4}\,yz]\\
&\quad + \frac{1}{\pi Ur^4}\left[\tfrac{1}{2}(z^2-y^2)\,(\delta_{i4}\,\delta_{j4} - \delta_{i3}\,\delta_{j3}) + yz\delta_{i3}\,\delta_{j4}\right].
\end{aligned}\right\} \tag{4.7}$$

These are seen to correspond to the Φ_0 contribution to the non-dissipative solution.

The Mach cone

For supersonic flow, the non-dissipative solution contains a characteristic surface known as the Mach cone. The solution is non-zero only in the region downstream of the cone, and becomes unbounded as the cone is approached. In the neighbourhood of the Mach cone, dissipation becomes important, and the solution must be re-examined in this region.

It is convenient in the present case to choose L to be some arbitrary macroscopic length, so that

$$\delta = \max\{\xi, \eta, \zeta\} \ll 1. \tag{4.8}$$

We shall obtain an asymptotic solution applicable for small δ. The final solution must actually be independent of L.

In (4.1) the cofactor matrix $\hat{\mathbf{C}}$ can again be removed from the integral as a differential operator, and the solution is given in terms of the integral,

$$\Phi = \frac{1}{(2\pi)^3}\iiint \frac{e^{i\mathbf{k}\cdot\mathbf{x}}\,dk_1\,dk_2\,dk_3}{D^*(ik_1, ik_2, ik_3)}, \tag{4.9}$$

where $D^* = D/(U^2k_1^2)$. From this it is evident that Φ is the solution to the differential equation,

$$D^*\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Phi = \delta(\mathbf{x}), \tag{4.10}$$

where $\delta(\mathbf{x}) = \delta(x)\,\delta(y)\,\delta(z)$. Expanding the determinant to obtain D, we find this equation is:

$$\left\{-U(U^2-\gamma)\frac{\partial^3}{\partial x^3}+\gamma U\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\delta\left[a\frac{\partial^4}{\partial x^4}+b\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+c\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)^2\right]\right.$$
$$\left.+O(\delta^2)\right\}\Phi = \delta(x)\,\delta(y)\,\delta(z). \tag{4.11}$$

The symbol $O(\delta^2)$ denotes additional terms of the form δ^{2+n} times derivatives of order $5+n$, where $n = 0, 1, 2$, and

$$a = \frac{\zeta}{\delta}(3U^2-2\gamma)+\frac{\eta}{\delta}U^2+\frac{\xi}{\delta}(U^2-1),$$

$$b = \frac{\zeta}{\delta}(3U^2-4\gamma)+\frac{\eta}{\delta}U^2+\frac{\xi}{\delta}(U^2-2),$$

$$c = -2\gamma\frac{\zeta}{\delta}-\frac{\xi}{\delta}.$$

If $\delta = 0$, the solution to (4.11) is

$$\Phi = -\frac{1}{2\pi\gamma U}\cosh^{-1}\frac{\alpha_h x}{(y^2+z^2)^{\frac{1}{2}}}, \tag{4.12}$$

where $\alpha_h^2 = \gamma/(U^2-\gamma)$. This result was obtained in §3. It is not a uniformly valid approximation for small δ, and the neighbourhood of the cone $\alpha_h^2x^2-(y^2+z^2) = 0$, $x \geq 0$ must be re-examined. To do this, we rewrite (4.11) in cylindrical co-ordinates, $r = (y^2+z^2)^{\frac{1}{2}}$, $\theta = \tan^{-1}(z/y)$, x, and then transform to the co-ordinates

(h, t, θ), where h denotes distance along the cone and t distance normal to it. This transformation is given by:

$$h = x\cos\theta_m + r\sin\theta_m,$$
$$t = -x\sin\theta_m = r\cos\theta_m.$$

where θ_m is the Mach cone angle, measured from the positive x axis. We define the inner variable $T = t/\sqrt{\delta}$, and consider an inner expansion for Φ of the form,

$$\Phi(h, T; \delta) = \Delta_1(\delta)\Phi_1(h, T) + o(\Delta_1), \tag{4.13}$$

where $\Delta_1(\delta)$ is to be determined. If the expansion (4.13) is substituted into (4.11), which must be rewritten in terms of h, T, θ, and the limit $\delta \to 0$ taken, we obtain the equation,

$$\left(\frac{\partial}{\partial h} + \frac{1}{2h} - \frac{s}{2\gamma\cos\theta}\frac{\partial^2}{\partial T^2}\right)\frac{\partial^2}{\partial T^2}\Phi_1 = 0, \tag{4.14}$$

where $s = a\sin^4\theta_m + b\sin^2\theta_m\cos^2\theta_m + c\cos^4\theta_m \geqslant 0$. The particular form of the stretching factor $\sqrt{\delta}$, used in defining the inner variable, was chosen to make the leading dissipative and non-dissipative terms the same order.

We need not consider the singularity on the right-hand side, since the appropriate solution to (4.14) will be obtained by matching with the outer solution, and by using certain integral conditions imposed by the requirement that the solution is the fundamental solution. We introduce the change in variables

$$\tau = (2\gamma\cos\theta_m/s)^{\frac{1}{2}}T \quad \text{and} \quad \psi = h^{\frac{1}{2}}\frac{\partial^2\Phi_1}{\partial\tau^2},$$

so that (4.14) becomes

$$\frac{\partial\psi}{\partial h} - \frac{\partial^2\psi}{\partial\tau^2} = 0. \tag{4.15}$$

The fundamental matrix is obtained by operating on Φ with the transposed cofactor matrix $\hat{\mathbf{C}}$. Since we have used D^* instead of D, we have

$$\mathbf{V} = \mathbf{C}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Phi,$$

where

$$\mathbf{C}(\mathbf{k}) = \hat{\mathbf{C}}(\mathbf{k})/U^2k_1^2.$$

Each of these operators must be written in terms of h, t, θ, transformed to inner variables, h, T, θ, and evaluated in the limit $\delta \to 0$. The result is that the fundamental matrix can be written in the form,

$$\mathbf{V} = -\frac{\Delta_1(\delta)}{\delta h^{\frac{1}{2}}}\mathbf{Y}\mathbf{Y}\psi, \tag{4.16}$$

where

$$\mathbf{Y} = \left(1, -\frac{\gamma}{U}, -\frac{(\gamma(U^2-\gamma))^{\frac{1}{2}}}{U}\cos\theta, -\frac{(\gamma(U^2-\gamma))^{\frac{1}{2}}}{U}\sin\theta, (\gamma-1)^{\frac{1}{2}}\right). \tag{4.17}$$

ψ must be determined as a solution of (4.15) such that it matches with the inviscid solution as $\tau \to -\infty$. To this end, we obtain the leading term in the inner expansion of the outer, inviscid solution. It is sufficient to match any one element of V_{ij}, say V_{11}, to obtain ψ and $\Delta_1(\delta)$. The matching of the remaining elements will then follow directly. From § 3, the outer solution for V_{11} is:

$$V_{11} = -\frac{1}{2\Pi}\frac{U}{\gamma^2}\frac{x}{[x^2-(1/\alpha_h^2)(y^2+z^2)]^{\frac{3}{2}}},$$

where $\alpha_h^2 = \gamma/(U^2-\gamma)$. We transform this solution to the (h,τ) variables, and take the leading term in the expansion in δ. The result is:

$$V_{11} \sim \frac{B}{\delta^{\frac{3}{4}}h^{\frac{1}{2}}(-\tau)^{\frac{3}{2}}} \quad (\tau < 0),$$

where
$$B = -\frac{U\cos^{\frac{7}{4}}\theta}{2^{\frac{7}{4}}\pi s^{\frac{3}{4}}\gamma^{\frac{1}{2}}(U^2-\gamma)^{\frac{3}{4}}}.$$

The expansion is limited to $\tau < 0$, corresponding to the interior of the Mach cone, since the non-dissipative solution vanishes outside the cone.

We must match this with the leading term of the outer expansion of the inner solution, which is given by:

$$V_{11} = -\frac{\Delta_1(\delta)}{\delta h^{\frac{1}{2}}}\psi_1(-\tau),$$

where $\psi_1(-\tau)$ denotes the leading term in the expansion of $\psi(\tau)$ for small δ (or large τ), with $t < 0$. Hence we see that $\Delta_1(\delta) = \delta^{\frac{1}{4}}$, and ψ must be such that

$$\psi(\tau) \sim \psi_1(-\tau) = -\frac{B}{|\tau|^{\frac{3}{2}}} \quad \text{as} \quad \tau \to -\infty. \tag{4.18}$$

Equation (4.15) admits a similarity solution of the form,

$$\psi = h^n g\left(-\frac{\tau^2}{4h}\right) = h^n g(\nu). \tag{4.19}$$

If ψ is to have the form (4.18) as $\tau \to -\infty$, or $\nu \to -\infty$, then clearly we must have:

$$g \sim -\frac{B}{2\sqrt{2}|\nu|^{\frac{3}{4}}} \quad \text{as} \quad \nu \to -\infty, \tag{4.20}$$

and $n = -\frac{3}{4}$. With this choice of n, g satisfies the confluent hypergeometric equation,

$$\nu g'' + (\tfrac{1}{2}-\nu)g' - \tfrac{3}{4}g = 0. \tag{4.21}$$

Two independent solutions are

$$\begin{aligned} g_1 &= H(\tfrac{3}{4},\tfrac{1}{2},\nu) \\ &= 1-\frac{3}{8}\frac{\tau^2}{h}+\frac{7}{128}\frac{\tau^4}{h^2}+\dots, \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} g_2 &= i\sqrt{\nu}\,H(\tfrac{5}{4},\tfrac{3}{2},\nu) \\ &= -\frac{\tau}{2\sqrt{h}}\left(1-\frac{5}{24}\frac{\tau^2}{h}+\frac{9}{384}\frac{\tau^4}{h^2}+\dots\right), \end{aligned} \tag{4.23}$$

where $H(a, c, x)$ is the confluent hypergeometric series. As $\nu \to -\infty$, we have

$$g_1 \to \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{4})} \frac{2\sqrt{2}\, h^{\frac{3}{4}}}{|\tau|^{\frac{3}{2}}} \quad (\tau \to \pm\infty), \tag{4.24}$$

$$g_2 \to -\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{4})} \frac{2\sqrt{2}\, h^{\frac{3}{4}}}{|\tau|^{\frac{3}{2}}} \operatorname{sgn}(\tau) \quad (\tau \to \pm\infty), \tag{4.25}$$

where $\Gamma(x)$ is the gamma function, and $\operatorname{sgn}(\tau)$ denotes the sign of τ. The full solution is a linear combination of g_1 and g_2,

$$g = c_1 g_1 + c_2 g_2,$$

where c_1, c_2 must be determined. One relation for c_1, c_2 is obtained from the matching condition (4.18), which, with (4.24) and (4.25), gives:

$$c_1 \frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{4})} + c_2 \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{4})} = -\frac{B}{2\sqrt{2}}. \tag{4.26}$$

In order to find c_1 and c_2, we need a second condition on g. This can be obtained by recalling that the solution must be a fundamental solution, and the governing equations contain the delta function on the right-hand side. Consider, e.g. the continuity equation,

$$\mathbf{U} . \nabla \rho + \nabla . \mathbf{u} = \delta(\mathbf{x}).$$

From this we obtain:

$$\int_S (\rho \mathbf{U} . \mathbf{n} + \mathbf{u} . \mathbf{n})\, dS = 1, \tag{4.27}$$

for any surface S enclosing the origin.

We use for ρ and $\mathbf{u}$ in this integral the composite expansion $v^c = v^{(o)} + v^{(i)} - v^{i(o)}$, which is a uniformly valid solution over the entire flow field. Here v denotes any variable, $v^{(o)}$ the outer solution, $v^{(i)}$ the inner solution, and $v^{i(o)}$ the inner expansion of the outer solution.

We substitute ρ^c and $\mathbf{u}^c$ into (4.27). Then, since†

$$\int_S (\rho^{(o)} \mathbf{U} . \mathbf{n} + \mathbf{u}^{(o)} . \mathbf{n})\, dS = 1,$$

we have:

$$\int_S (\rho^{(i)} \mathbf{U} . \mathbf{n} + \mathbf{u}^{(i)} . \mathbf{n})\, dS = \int_S (\rho^{i(o)} \mathbf{U} . \mathbf{n} + \mathbf{u}^{i(o)} . \mathbf{n})\, dS. \tag{4.28}$$

We choose for S the conical surface S with apex at $(x_0, 0, 0)$, normal to the Mach cone, and extending to infinity, i.e. the surface,

$$(x - x_0)^2 - \alpha_h^2 (y^2 + z^2) = 0 \quad (x < x_0).$$

More precisely, we consider the finite portion of S given by $x_1 < x < x_0$, and the hemisphere,

$$(x - x_1)^2 + y^2 + z^2 = (1/\alpha_h^2)(x_1 - x_0)^2 \quad (x < x_1),$$

† This integral does not, in fact, exist in the ordinary sense, since the integrand becomes infinite as $1/t^{\frac{3}{2}}$ as the Mach cone is approached. However, fundamental solutions must be expressed as distributions; it is in this sense that such integrals must be evaluated (see e.g. Gelfand & Shilov 1964). Alternatively, one may use the concept of finite part, as first introduced by Hadamard (1932).

and take the limit $x_1 \to -\infty$. The integral over the hemisphere vanishes in the limit, and we are left only with the integral over S.

On the surface S, $U.n = U\cos\theta_m$ and $u.n = u_x \cos\theta_m + (u_y^2 + u_z^2)^{\frac{1}{2}} \sin\theta_m$. In terms of the fundamental matrix the integral in (4.28) is

$$\sum_j \int_S [V_{1j} U \cos\theta_m + V_{2j} \cos\theta_m + (V_{3j}^2 + V_{4j}^2)^{\frac{1}{2}} \sin\theta_m] \, dS.$$

The left-hand side is obtained by using

$$V_{ij}^{(i)} = -\frac{1}{\delta^{\frac{3}{4}} h^{\frac{1}{2}}} Y_i Y_j \psi,$$

and the right-hand side by using

$$V_{ij}^{i(o)} = \frac{B}{\delta^{\frac{3}{4}} h^{\frac{1}{2}} (-\tau)^{\frac{3}{2}}} Y_i Y_j.$$

(We obtained above the form of $V_{11}^{i(o)}$. The remaining elements of $V_{ij}^{i(o)}$ are found in the same manner.)

Equation (4.28) reduces to

$$\int_S \psi \, dS = -B \int_S \frac{1}{(-\tau)^{\frac{3}{2}}} dS. \tag{4.29}$$

Now,
$$\int_S \frac{1}{(-\tau)^{\frac{3}{2}}} \, dS = \int_0^{x_0 \sin\theta_m} \frac{2\pi R \, dR}{(-\tau)^{\frac{3}{2}}},$$

where R is distance along S from the apex $(x_0, 0, 0)$, and is related to τ by $R = x_0 \sin\theta_m + t = x_0 \sin\theta_m - \lambda\delta^{\frac{1}{2}}\tau' (\lambda = (s/2\gamma \cos\theta_m)^{\frac{1}{2}}, \tau' = -\tau)$. Hence, the above integral becomes:

$$2\pi \int_0^{x_0 \sin\theta_m / \sqrt{\delta}\,\lambda} \frac{x_0 \sin\theta_m - \lambda\sqrt{\delta}\,\tau'}{\tau'^{\frac{3}{2}}} \, d\tau'.$$

This integral, which must be interpreted within the framework of the theory of distributions (see the last footnote), vanishes to lowest order in δ. Consequently, ψ must satisfy the condition

$$\int_S \psi \, dS = 0, \tag{4.30}$$

or
$$\int_0^\infty g^*(\tau/\sqrt{h}) \, R \, dR = 0,$$

where $g^*(\tau/\sqrt{h}) = g^*(\sigma) = g(\tau^2/h)$. Now, $R = x_0 \sin\theta - \lambda(\delta h)^{\frac{1}{2}} \sigma$, so the above integral may be written:

$$\int_{-\infty}^{x_0 \sin\theta / \lambda(\delta h)^{\frac{1}{2}}} g^*(\sigma) \, (x_0 \sin\theta_m - \lambda(\delta h)^{\frac{1}{2}} \sigma) \, \lambda(\delta h)^{\frac{1}{2}} \, d\sigma.$$

To lowest order in δ, we therefore have

$$\int_{-\infty}^{+\infty} g^*(\sigma) \, d\sigma = 0.$$

This condition requires that g^* be an odd function of τ. Hence $c_1 = 0$, and the solution is

$$\psi(\tau, h) = \frac{B\tau}{4\sqrt{2}\, h^{\frac{5}{4}}} \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{2})} H\left(\tfrac{5}{4}, \tfrac{3}{2}, -\frac{\tau^2}{4h}\right), \tag{4.31}$$

or

$$\mathbf{V} = 0{\cdot}191 \frac{U \cos^{\frac{7}{4}}\theta}{\delta^{\frac{3}{4}}\gamma^{\frac{1}{2}}(U^2-\gamma)\, s^{\frac{3}{4}}h^{\frac{5}{4}}} \mathbf{YY} L\left(\frac{\tau}{\sqrt{h}}\right), \tag{4.32}$$

where $L(\tau/\sqrt{h}) = \tau/\sqrt{h}\, H(\frac{5}{4}, \frac{3}{2}, -\tau^2/4h)$ is plotted in figure 1. Note that

$$L(-x) = -L(x).$$

As $x \to \infty$, $L(x) \to 1{\cdot}382/x^{\frac{3}{2}}$. This curve is indicated by the broken line in the figure.

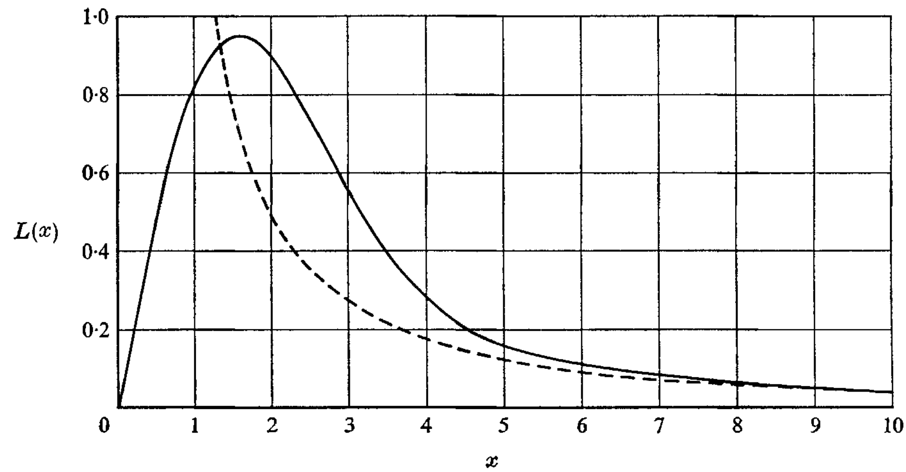

FIGURE 1. The function $L(\tau/\sqrt{h})$, which describes the structure of the Mach cone according to (4.32). The curve is antisymmetric, $L(-\tau/\sqrt{h}) = -L(\tau/\sqrt{h})$, and is plotted only for positive argument. The broken curve represents the inviscid solution.

5. Conclusions

The fundamental solution provides the far field flow within the framework of the approximations described above. The various regions which we have studied can be combined to obtain a picture of the complete flow field. Behind the body there is a wake structured by the dissipative effects. In subsonic flow the effects of viscosity and thermal conductivity are confined to this region, but in supersonic flow the Mach cone also becomes structured through the action of dissipation. Behind the Mach cone, the flow is given by the non-dissipative theory, and this solution matches the solution near the Mach cone in an asymptotic sense.

The solution in the wake is a natural extension of the results of Sirovich (1968). There the wake was found to be of the form $e^{-\lambda y^2/x}/\sqrt{x}$, where y is the direction normal to the wake. In the present case, this term is replaced by

$$(e^{-\lambda y^2/x}/\sqrt{x})\,(e^{-\lambda z^2/x}/\sqrt{x}).$$

In addition, there are terms with no counterpart in the two-dimensional case (V_{33}, V_{34}, V_{44}), which describe the cross-flow.

The wake solution is a generalization to compressible flow of the well-known solution of the far field laminar wake (e.g. Landau & Lifshitz 1959), and is completely analogous in form. However, the wake now consists of two parts, one a vorticity wake structured by viscosity, and the other an entropy wake structured by thermal conductivity. Only the vorticity wake occurs in incompressible flow. This decoupling of the wake was also noted by Sirovich in the two-dimensional case.

The structure of the three-dimensional Mach cone is in complete contrast to the two-dimensional Mach lines. The solution for the Mach lines is the same as for the wake, with y and x replaced by distance normal to, and distance along, the Mach line (Sirovich 1968). In the three-dimensional case, the solution grows like $1/t^{\frac{3}{2}}$ as the Mach cone is approached, as in the non-dissipative theory. However, as t approaches zero, the dissipation becomes important, causing the solution to decrease, pass through the origin at $t = 0$, and then behave in an antisymmetric fashion for $t < 0$, as illustrated in figure 1.

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